Converting separable conditions to entanglement breaking conditions

Ryo Namiki
Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
(Dated: March 25, 2015)

We present a general method to derive entanglement breaking (EB) conditions for continuous-variable quantum gates. We start with an arbitrary entanglement witness, and reach an EB condition. The resultant EB condition is applicable not only for quantum channels but also for general quantum operations, namely, trace-non-increasing class of completely positive maps. We illustrate our method associated with a quantum benchmark based on the input ensemble of Gaussian distributed coherent states. We also exploit our idea for channels acting on finite dimensional systems and present a Schmidt-number benchmark based on input states of two mutually unbiased bases and measurements of generalized Pauli operators.

arXiv:1503.07109v1 [quant-ph] 24 Mar 2015

An important task for future realization of quantum information technology is to establish a reliable quantum channel. A powerful tool to estimate an experimental implementation of quantum gates is quantum process tomography. However, it is not always feasible to measure the input-and-output relations for a set of tomographic complete states. Instead of tomographic approach, one may be interested in probing a basic coherence of quantum channels using a small set of feasible input states. The quantum benchmarks provide such a method based on the context of quantum entanglement [1–7]. A quantum benchmark is typically determined by an upper bound of an average fidelity achieved by a class of quantum channels called entanglement breaking (EB) [8]. If an experimental fidelity surpasses the fidelity bound we can verify that any classical measure-and-prepare map is unable to simulate the channel. Mathematically, it implies the Choi-Jamiolkowsky (CJ) state of the channel is entangled, hence, there exists, at least, one entangled input state whose inseparability maintains under the channel action. There have been several works to determine such classical fidelities [3–5, 9–11] or other forms of EB limits [12, 13]. One can also apply the notion of EB limits to quantum operations, namely, trace-non-increasing class of completely positive (CP) maps [14, 15]. In addition to a proof of the inseparability in the physical process, one can demonstrate a more specified type of channel's coherence by quantifying the amount of entanglement in the CJ state [16–18].

Although it has been known that an EB condition is mathematically equivalent to a separable condition, the varieties of known EB conditions are rather limited compared with those of known separable conditions. In fact, one can easily find several systematic methods to produce a series of separable conditions [19–21] whereas potential applications of separable conditions to the quantum benchmark problems have little been mentioned in the literatures on the separability problems [22, 23].

In this report, we present a general method to convert a separable condition to an EB condition for continuous-variable quantum channels as a generalization of the method developed in [13]. Given a formula of entanglement witness we compose an EB condition by separately assigning an entangled density operator. After a general composition we illustrate our method associated with a quantum benchmark based on the Gaussian distributed coherent states [15]. We also exploit our idea for channels acting on finite dimensional systems and present a Schmidt-number benchmark [16] for qudit channels based on input states of two mutually unbiased bases and measurements of generalized Pauli operators.

Let ρ be a density operator and write an expectation value of an operator $\hat{O}$ as $\mathrm{Tr}[\hat{O}\rho] = \langle \hat{O}\rangle_\rho$. A general form of separable conditions can be written by a function of expectation values for a set of operators $\{\hat{O}_i\}_{i=1,2,\cdots,N}$ as

$$F(\langle \hat{O}_1\rangle_\rho, \langle \hat{O}_2\rangle_\rho, \cdots, \langle \hat{O}_N\rangle_\rho) \geq 0. \tag{1}$$

A special case is based on an operator called the *entanglement witness* $\hat{W}$ that satisfies

$$\mathrm{Tr}(\hat{W}\rho_s) = \langle \hat{W}\rangle_{\rho_s} \geq 0 \tag{2}$$

for any separable state $\rho_s = \sum p_i (|a_i\rangle\langle a_i|)_A \otimes (|b_i\rangle\langle b_i|)_B$. It implies that ρ is entangled when $\langle \hat{W}\rangle_\rho < 0$ holds. In what follows we derive an EB condition starting from an witness operator $\hat{W}$. We can readily extend our method for the general form in Eq. (1). This form includes non-linear terms of expectation values and is often referred to as the *non-linear witness*.

We consider a two-mode system AB described by bosonic field operators satisfying the commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$. Let us suppose $\hat{W}$ is expressed in the anti-normal order regarding to the field operators $\{b, b^\dagger\}$ for the second system B such as

$$\hat{W}(a, a^\dagger, b, b^\dagger) = \sum_{n,m} W^{(n,m)}(a, a^\dagger) b^n (b^\dagger)^m. \tag{3}$$

Then, we can rewrite it as

$$\begin{aligned}\hat{W} &= \sum W_{n,m}(a, a^\dagger) b^n \mathbb{1}_B (b^\dagger)^m \\ &= \sum W^{(n,m)}(a, a^\dagger) \int (\alpha^*)^n \alpha^m |\alpha^*\rangle\langle\alpha^*| \frac{d^2\alpha}{\pi} \\ &= \int \hat{W}(a, a^\dagger, \alpha^*, \alpha) |\alpha^*\rangle\langle\alpha^*| \frac{d^2\alpha}{\pi},\end{aligned} \tag{4}$$

where we used the closure relation for coherent states $\int |\alpha\rangle\langle\alpha| d^2\alpha/\pi = \mathbb{1}$ for the subsystem B. Here, we express the closure with α^*, the complex conjugate of α, for a notation convention. Equation (4) implies

$$\mathrm{Tr}(\hat{W}\rho) = \mathrm{Tr}_A\left[\int \hat{W}(a, a^\dagger, \alpha^*, \alpha)\langle\alpha^*|\,\rho\,|\alpha^*\rangle_B \frac{d^2\alpha}{\pi}\right], \tag{5}$$

where Tr_A denotes partial trace over system A.

Let $\psi = \psi_{AB}$ be an entangled density operator of the two-mode field AB. We define an ensemble of states $\{p_\alpha, \varphi_\alpha\}$ on a one-mode system as

$$\begin{aligned} p_\alpha &:= \mathrm{Tr}\left[\mathbb{1}_A \otimes (|\alpha^*\rangle\langle\alpha^*|)_B \psi_{AB}\right], \\ \varphi_\alpha &:= \langle\alpha^*|\,\psi_{AB}\,|\alpha^*\rangle_B / p_\alpha. \end{aligned} \tag{6}$$

Note that φ_α is a type of the relative states of $|\alpha^*\rangle$ regarding ψ_{AB} and p_α is a probability density satisfying $\int p_\alpha d^2\alpha/\pi = 1$.

Let us consider the local action of a physical map $\mathcal{E}$ for the state ψ,

$$J = \mathcal{E}_A \otimes I_B(\psi) \tag{7}$$

where $\mathcal{E}$ is a CP map acting on system A and I is the identity map. When $\mathcal{E}$ is a trace-decreasing operation, we can formally normalize J as a density operator by J/P_s with

$$P_s := \mathrm{Tr}[J] = \int p_\alpha \mathrm{Tr}[\mathcal{E}(\varphi_\alpha)] d^2\alpha/\pi, \tag{8}$$

where we use the relations in Eq. (6). Note that we have $P_s = 1$ for the trace-preserving maps. Substituting $\rho = J/P_s$ into Eq. (5) we can write

$$\mathrm{Tr}(\hat{W}\rho) = \frac{1}{P_s}\mathrm{Tr}\left[\int \hat{W}(a, a^\dagger, \alpha^*, \alpha) p_\alpha \mathcal{E}(\varphi_\alpha)\frac{d^2\alpha}{\pi}\right]. \tag{9}$$

Here, system B is traced out and $\mathrm{Tr}(\hat{W}\rho)$ is represented by the mean values of operators on system A over channel's outputs $\mathcal{E}(\varphi_\alpha)$ subjected to the input state $\{\varphi_\alpha\}$.

Let us suppose that $\mathcal{E}$ is an EB map, i.e., $\mathcal{E}(\rho) = \sum_i \mathrm{Tr}[M_i\rho]\sigma_i$ with $M_i \geq 0$, $\sum_i M_i \leq \mathbb{1}$, and a set of density operators $\{\sigma_i\}$. Then, ρ becomes a separable density operator and $\mathrm{Tr}(\hat{W}\rho)$ has to fulfills the separable condition of Eq. (2). Therefore, we obtain the following EB condition:

$$\frac{1}{P_s}\mathrm{Tr}\left[\int \hat{W}(a, a^\dagger, \alpha^*, \alpha) p_\alpha \mathcal{E}(\varphi_\alpha)\frac{d^2\alpha}{\pi}\right] \geq 0. \tag{10}$$

In this manner one can compose an EB condition from a separable condition by separately assigning an entangled state ψ. To be concrete, the inequality of Eq. (10) is a necessary condition for entanglement breaking, and any violation of this inequality implies that the map $\mathcal{E}$ cannot be an EB map.

For a non-linear witness in the form of Eq. (1), we simply assign an operators $\hat{W}_i$ for each of $\hat{O}_i$ and express its expectation value as in Eq. (9) by repeating the procedure above. Then, we can generally convert separable conditions in the form of Eq. (1) into EB conditions by replacing the relevant expectation values as follows:

$$\langle\hat{O}_i\rangle_\rho \to \frac{1}{P_s}\mathrm{Tr}\left[\int \hat{W}_i(a, a^\dagger, \alpha^*, \alpha) p_\alpha \mathcal{E}(\varphi_\alpha)\frac{d^2\alpha}{\pi}\right]. \tag{11}$$

Note that the obtained EB condition depends on the choice of the entanglement ψ which determines the state ensemble $\{p_\alpha, \varphi_\alpha\}$ owing to Eq. (6). Accordingly, a different choice of ψ could constitute a different EB condition even the original separable condition is the same.

Let us illustrate our method associated with a familiar case of the fidelity-based quantum benchmark [4, 5, 11, 15]. In experiments of quantum optics, coherent states are commonly available as a state of laser light. It is thus feasible to probe an experimental quantum gate by an input of coherent states. We will consider an input ensemble of the Gaussian distributed coherent states [24]. This ensemble can be associated with the case that ψ is a two-mode squeezed state. In fact, by substituting the two-mode squeezed state $|\psi_\xi\rangle = \sqrt{1-\xi^2}\sum_{n=0}^\infty \xi^n |n\rangle|n\rangle$ with $\xi \in (0,1)$ into Eqs. (6), we obtain the ensemble of Gaussian distributed coherent states,

$$\begin{aligned} p_\alpha &= (1-\xi^2)e^{-(1-\xi^2)|\alpha|^2}, \\ \varphi_\alpha &= |\xi\alpha\rangle\langle\xi\alpha|. \end{aligned} \tag{12}$$

Let $X \geq 0$ and (u, v) be a pair of real number that fulfills $u^2 + v^2 = 1$ and $u \neq 0$. Let us define an witness operator

$$\hat{W} := \frac{\mathbb{1}}{1+X} - \frac{1}{\pi}\int e^{-X|\alpha|^2}|v\alpha\rangle\langle v\alpha| \otimes |u\alpha^*\rangle\langle u\alpha^*| d^2\alpha, \tag{13}$$

such that $\langle\hat{W}\rangle \geq 0$ becomes the separable condition in Eq. (21) of Ref. [25]. Since $\hat{W}$ is already expanded in the local coherent states similar to the form in Eq. (4) it is no need to consider the operator ordering. From Eqs. (9), (12), and (13) we can write

$$\begin{aligned} \mathrm{Tr}[\hat{W}J] = &\frac{1}{1+X} - \frac{1}{\pi P_s u^2}\left(\lambda + \frac{X}{\xi^2 u^2}\right) \\ &\times \int e^{-\lambda|\alpha|^2}\langle\sqrt{\eta}\alpha|\,\mathcal{E}(|\alpha\rangle\langle\alpha|)\,|\sqrt{\eta}\alpha\rangle d^2\alpha, \end{aligned} \tag{14}$$

where $\lambda = \xi^{-2}(Xu^{-2} + (1-\xi^2))$ and $\eta := v^2/(\xi u)^2$. Using the condition of Eq. (10) and taking the limit $X \to 0$ we obtain the following EB condition

$$P_s - \frac{1}{u^2}\frac{\lambda}{\pi}\int e^{-\lambda|\alpha|^2}\langle\sqrt{\eta}\alpha|\,\mathcal{E}(|\alpha\rangle\langle\alpha|)\,|\sqrt{\eta}\alpha\rangle d^2\alpha \geq 0, \tag{15}$$

where $u^2 = (1+\lambda+\eta)/(1+\lambda)$. This corresponds to the fidelity-based quantum benchmark for general CP maps [15]. In Ref. [15], its derivation is based on the duality of semidefinite programing. For quantum channels (the trace-preserving class of CP maps; $P_s = 1$), one can find other derivations in Refs. [4, 5, 11].

Note that there is a wide interest in formulating separable conditions based on the moments of canonical quadrature variables [19, 26–28]. The moments of quadrature variables can be directly observed by homodyne measurements in experiments. Among all, second-order conditions have been widely used as a feasible method for entanglement detection. It is well-known that the sum condition [26] and the product condition [28] are sufficient for witnessing two-mode Gaussian entanglement. By applying our method we can translate them into the EB conditions with the input ensemble of the Gaussian distributed coherent states in [13] (Corollary 1 and Proposition, respectively), which are sufficient to witness one-mode Gaussian channels in the quantum domain, namely, one-mode Gaussian channels being nonmember of the EB class. Further, the formalism developed in Ref. [13] would be usable as a quantitative quantum benchmark because it can be related to entanglement of formation on the CJ state (See Ref. [29]). Similar statements could hold for the fidelity-based approach. In fact, the entanglement witness of Eq. (13) is known to be sufficient for witnessing two-mode Gaussian entanglement [25] and the fidelity-based EB condition is also sufficient for detecting one-mode Gaussian channels in the quantum domain [5]. However, its connection to a meaningful entanglement measure is left open.

In the rest of this report, we discuss the case of the physical process acting of a finite dimensional system. The key mechanism to introduce the ensemble of input states $\{p_\alpha, \varphi_\alpha\}$ in Eq. (6) is the coherent-state expression of system B in Eq. (4). Analogously, we can introduce a state ensemble by decomposing the witness operator with a set of hermitian operators $\hat{h}$ on system B as follows

$$\hat{W} = \sum_l w_A^{(l)} \otimes \hat{h}_B^{(l)} = \sum_l w_A^{(l)} \otimes \left(\sum_j h_j{}^{(l)} |j^{(l)}\rangle\langle j^{(l)}| \right)_B, \tag{16}$$

where $\{h_j^{(l)}, |j^{(l)}\rangle\}$ represents the spectral decomposition of $\hat{h}^{(l)}$. This implies the set of input states similarly to Eq. (6) as

$$\begin{aligned} p_{j,l} &:= \mathrm{Tr}\left[\mathbb{1}_A \otimes (|j^{(l)}\rangle\langle j^{(l)}|)_B \psi_{AB} \right] \\ \varphi_{j,l} &:= \langle j^{(l)}|\psi_{AB}|j^{(l)}\rangle_B / p_{j,l}. \end{aligned} \tag{17}$$

Therefore, instead of Eq. (10), we obtain an EB condition in the following form:

$$\frac{1}{P_s} \sum_{j,l} p_{i,l} h_j^{(l)} \mathrm{Tr}\left[\hat{w}^{(l)} \mathcal{E}(\varphi_{j,l}) \right] \geq 0, \tag{18}$$

where we define $P_s = \sum_{j,l} \mathrm{Tr}[p_{j,l}\mathcal{E}(\varphi_{j,l})]$. Note that an example of the decomposition in Eq. (16) can be obtained by choosing a Hilbert-Schmidt orthonormal basis on subsystem B.

Finally, using this framework we will derive a Schmidt number benchmark [16] for quantum operations acting on a d-dimensional (qudit) system. The Schmidt number benchmark of class $k+1$ ($k \in [1, d-1]$) enables us to eliminate the possibility that the channel is described by Kraus operators of rank k or less than k. This class of quantum channels is called k-partial EB channels [30–32], and $k = 1$ represents the class of EB channels.

Let us consider a Schmidt number-$(k+1)$ witnesse for two d-dimension system given in Ref. [33],

$$g_{k,d} - \frac{1}{2} \langle \hat{Z}_A \hat{Z}_B^\dagger + \hat{Z}_A^\dagger \hat{Z}_B + \hat{X}_A \hat{X}_B + \hat{X}_A^\dagger \hat{X}_B^\dagger \rangle \geq 0, \tag{19}$$

where $g_{k,d} = [(d-k)\cos\omega + (d+k)]/d$, and $\hat{Z} := \sum_{j=0}^{d-1} e^{i\omega j} |j\rangle\langle j|$ and $\hat{X} := \sum_{j=0}^{d-1} |j+1\rangle\langle j|$, are the generalized Pauli operators. Here, we assumed a fixed Z-basis $\{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$ with modulo-d conditions $|j+d\rangle = |j\rangle$ and $\omega := 2\pi/d$. By expanding $\hat{Z}$ and $\hat{X}$ respectively in Z-basis $\{|j\rangle\}$ and X-basis $\{|\bar{j}\rangle\}$, which is defined through $|\bar{l}\rangle := \hat{Z}^l \left(\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \right) = \hat{Z}^l |\bar{0}\rangle$, we can see that the operators on system B in Eq. (19) can be expressed by the projections onto the mutually unbiased bases, $\{|j\rangle, |\bar{j}\rangle\}$. Using this expansion and $J = \mathcal{E}_A \otimes I_B(|\Phi_d\rangle\langle\Phi_d|)/P_s$ with $|\Phi_d\rangle = d^{-1/2} \sum_{j=0}^{d-1} |j\rangle |j\rangle$ with Eqs. (17) and (18) we obtain the following necessary condition for k-partial EB maps:

$$\begin{aligned} P_s g_{k,d} - \sum_{j=0}^{d-1} \mathrm{Tr}\big[&(\hat{Z}e^{-i\omega j} + \hat{Z}^\dagger e^{i\omega j}) \mathcal{E}(|j\rangle\langle j|) \\ &+ (\hat{X}e^{-i\omega j} + \hat{X}^\dagger e^{i\omega j}) \mathcal{E}(|\overline{-j}\rangle\langle\overline{-j}|) \big]/d \geq 0. \end{aligned} \tag{20}$$

Hence, a violation of this condition implies a quantitative quantum benchmark for the Schmidt class $k+1$, namely, any Kraus representation of $\mathcal{E}$ has, at least, one Kraus operator whose rank is $k+1$ or higher. An experimental test would be executed by input states of two mutually unbiased bases and projections to these bases similarly to the result in Ref. [16]. Note that we can readily extend the result in Ref. [16] for quantum operations acting on qudit states by using the normalized state $J = \mathcal{E}_A \otimes I_B(|\Phi_d\rangle\langle\Phi_d|)/P_s$.

In summary, we have presented a method to convert separable conditions to EB conditions for bosonic single-mode channels. Given an entanglement witness we can generate an EB condition by separately assigning an entangled state that determines the ensemble of input states. By considering a normalization of this state the resultant EB condition becomes applicable to general quantum operations, namely, trace-non-increasing class of CP maps. As an example we present a different derivation of the fidelity-based quantum benchmark in Ref. [15] starting from a separable condition given in

Ref. [25]. Although we focus on single-mode operations, our method can be straightforwardly extended for multi-mode bosonic quantum channels/operations. We have also developed a similar framework for quantum operations acting on finite dimension systems and presented a Schmidt number benchmark for quantum operations.

Acknowledgments

RN was supported by the DARPA Quiness program under prime Contract No. W31P4Q-12-1-0017 and Industry Canada.

[1] S. Popescu, Phys. Rev. Lett. **72**, 797 (1994).
[2] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
[3] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).
[4] K. Hammerer, M. M. Wolf, E. S. Polzik, and J. I. Cirac, Phys. Rev. Lett. **94**, 150503 (2005).
[5] R. Namiki, M. Koashi, and N. Imoto, Phys. Rev. Lett. **101**, 100502 (2008).
[6] M. Owari, M. B. Plenio, E. S. Polzik, a. Serafini, and M. M. Wolf, New J. Phys. **10**, 113014 (2008).
[7] J. Calsamiglia, M. Aspachs, R. Muñoz Tapia, and E. Bagan, Phys. Rev. A **79**, 050301 (2009).
[8] M. B. Ruskai, Rev. Mod. Phys. **15**, 643 (2003).
[9] C. A. Fuchs and M. Sasaki, Quantum Inf. Comput. **3**, 377 (2003).
[10] R. Namiki, Phys. Rev. A **78**, 032333 (2008).
[11] R. Namiki, Phys. Rev. A **83**, 042323 (2011).
[12] H. Häseler and N. Lütkenhaus, Phys. Rev. A **80**, 042304 (2009).
[13] R. Namiki and K. Azuma, arXiv:1404.2643, To appear Phys. Rev. Lett. (2015).
[14] Y. Yang, G. Chiribella, and G. Adesso, Phys. Rev. A (2014).
[15] G. Chiribella and J. Xie, Phys. Rev. Lett. **110**, 213602 (2013).
[16] R. Namiki and Y. Tokunaga, Phys. Rev. A **85**, 010305 (2012).
[17] N. Killoran, M. Hosseini, B. C. Buchler, P. K. Lam, and N. Lütkenhaus, Phys. Rev. A **86**, 022331 (2012).
[18] I. Khan *et al.*, Phys. Rev. A **88**, 010302 (2013).
[19] E. Shchukin and W. Vogel, Phys. Rev. Lett. **95**, 230502 (2005).
[20] A. Miranowicz, M. Piani, P. Horodecki, and R. Horodecki, Phys. Rev. A **80**, 052303 (2009).
[21] A. Miranowicz, M. Bartkowiak, X. Wang, Y.-x. Liu, and F. Nori, Phys. Rev. A **82**, 013824 (2010).
[22] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[23] O. Gühne and G. Tóth, Physics Reports **474**, 1 (2009).
[24] S. Braunstein, C. Fuchs, and H. Kimble, J. Mod. Opt. **47**, 267 (2000).
[25] R. Namiki, J. Phys. Soc. Jpn. **82**, 14001 (2013).
[26] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[27] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[28] V. Giovannetti, S. Mancini, D. Vitali, and P. Tombesi, Phys. Rev. A **67**, 022320 (2003).
[29] R. Namiki, arXiv:1502.05031.
[30] D. Chruciski and A. Kossakowski, Open Systems & Information Dynamics **13**, 17 (2006).
[31] S. Huang, Phys. Rev. A **73**, 052318 (2006).
[32] R. Namiki, Phys. Rev. A **88**, 064301 (2013).
[33] R. Namiki and Y. Tokunaga, Phys. Rev. Lett. **108** (2012).